Problem set 4

Due date: 18th Feb

Part A (submit any three)

Exercise 28. (1) Let $X \geq 0$ be a r.v on $(\Omega, \mathcal{F}, \mathbf{P})$ with $0 < \mathbf{E}[X] < \infty$. Then, define $\mathbf{Q}(A) = \mathbf{E}[X\mathbf{1}_A]/\mathbf{E}[X]$ for any $A \in \mathcal{F}$. Show that $\mathbf{Q}$ is a probability measure on $\mathcal{F}$. Further, show that for any bounded random variable Y, we have $\mathbf{E}_{\mathbf{Q}}[Y] = \frac{\mathbf{E}[YX]}{\mathbf{E}[X]}$.

(2) If μ and ν are Borel probability measures on the line with continuous densities f and g (respectively) w.r.t. Lebesgue measure. Under what conditions can you assert that μ has a density w.r.t ν? In that case, what is that density?

Exercise 29. For $p = 1, 2, \infty$, check that $\|X - Y\|_p$ is a metric on the space $L^p := \{[X] : \|X\|_p < \infty\}$ (here $[X]$ denotes the equivalence class of X under the equivalence relation $X \sim Y$ if $\mathbf{P}(X = Y) = 1$).

Exercise 30. (1) If X is a non-negative r.v., show that $\mathbf{E}[X] = \int_0^\infty \mathbf{P}[X > t]dt$. What analogous formula holds for a general integrable r.v.?

(2) If X is a non-negative integer valued r.v., then $\mathbf{E}[X] = \sum_{n=1}^\infty \mathbf{P}(X \geq n)$.

Exercise 31. Show that the values $\mathbf{E}[f \circ X]$ as f varies over the class of all smooth (infinitely differentiable), compactly supported functions determine the distribution of X.

Exercise 32. (i) Express the mean and variance of of $aX + b$ in terms of the same quantities for X (a, b are constants).
(ii) Show that $\text{Var}(X) = \mathbf{E}[X^2] - \mathbf{E}[X]^2$.

Part B (submit any two)

Exercise 33. Compute mean, variance and moments (as many as possible!) of the Normal(0,1), exponential(1), Beta(p,q) distributions.

Exercise 34. (1) If X_n are non-negative r.v. and $X_n \downarrow X$, and $\mathbf{E}[X_n] < \infty$ for some n, then show that $\mathbf{E}[X_n] \to \mathbf{E}[X]$.

(2) If $\mathbf{E}[|X|] < \infty$, then $\mathbf{E}[|X|\mathbf{1}_{|X|>A}] \to 0$ as $A \to \infty$.

Exercise 35. (1) Suppose (X, Y) has a continuous density $f(x, y)$. Find the density of X/Y. Apply to the case when (X, Y) has the *standard bivariate normal distribution* with density $f(x, y) = (2\pi)^{-1} \exp\{-\frac{x^2+y^2}{2}\}$.

(2) Find the distribution of $X + Y$ if (X, Y) has the standard bivariate normal distribution.

(3) Let $U = \min\{X, Y\}$ and $V = \max\{X, Y\}$. Find the density of (U, V).

Exercise 36. Let $\mu_n, \mu \in \mathcal{P}(\mathbb{R}^n)$. Show that $\mu_n \xrightarrow{d} \mu$ if and only if $\int f d\mu_n \to \int f d\mu$ for every $f \in C_b(\mathbb{R})$.